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A Localization of a Semigroup Ring,II

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This is a continuation of our [M4]. Thus a submonoid S of a torsion-free abelian (additive) group is called a g-monoid. For a g-monoid S, the quotient group of S is denoted by $q(S)$, and for a commutative ring R, the total quotient ring of R is denoted by $q(R)$. Throughout the paper S denotes a g-monoid which is not $\{0\}$.

Let $F(S)$ be the set of fractional ideals of the g-monoid S. A mapping $I \longmapsto I^*$ of $F(S)$ to $F(S)$ is called a star-operation on S if the following conditions hold for every element $a \in q(S)$ and $I, J \in F(S)$:

$(a)^* = (a)$; $(a+I)^* = a + I^*$; $I \subset I^*$;

If $I \subset J$, then $I^* \subset J^*$; $(I^*)^* = I^*$.

Let $*$ be a star-operation on S. If, for all finitely generated fractional ideals J_1, J_2 and I, $(I+J_1)^* \subset (I+J_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an e.a.b. star-operation on S.

Let $F'(S)$ be the set of non-empty subsets of $q(S)$ such that $S + I \subset I$. A mapping $I \longmapsto I^*$ of $F'(S)$ to $F'(S)$ is called a semistar-operation on S if the following conditions hold for every element $a \in q(S)$ and $I, J \in F'(S)$:

$(a+I)^* = a + I^*$; $I \subset I^*$;

If $I \subset J$, then $I^* \subset J^*$; $(I^*)^* = I^*$.

Let $*$ be a semistar-operation on S. If, for all finitely genarated fractional ideals J_1, J_2 and I, $(I+J_1)^* \subset (I+J_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an e.a.b. semistar-operation on S.

Let R be a commutative ring. A non-zerodivisor of R is called a regular element of R. If an ideal I of R contains at least one regular element, then I is called a regular ideal of R. If every regular ideal is generated by regular elements, then R is called a Marot ring. If, for every regular element f of the polynomial ring $R[X]$, the ideal of R generated by the coefficients of f is a regular ideal of R, then R is said to have property (A).

Let I be an R-submodule of $q(R)$ such that $rI \subset R$ for some regular $r \in R$. Then I is called a fractional ideal of R. Let $F(R)$ be the set of non-zero fractional ideals of R. A mapping $I \longmapsto I^*$ of $F(R)$ to $F(R)$ is called a star-operation on R if the following conditions hold for every regular element $a \in q(R)$ and $I, J \in F(R)$:

(1) $(a)^* = (a)$; (2) $(aI)^* = aI^*$; (3) $I \subset I^*$;

(4) If $I \subset J$, then $I^* \subset J^*$; (5) $(I^*)^* = I^*$.

Let $*$ be a star-operation on R. If, for all finitely generated non-zero fractional ideals J_1, J_2, I with I regular, $(IJ_1)^* \subset (IJ_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an e.a.b. star-operation on R.

Let $F'(R)$ be the set of non-zero R-submodules of $q(R)$. A mapping $I \longmapsto I^*$ of $F'(R)$ to $F'(R)$ is called a semistar-operation on R if the following conditions hold for every regular element $a \in q(R)$ and $I, J \in F'(R)$:

(1) $(aI)^* = aI^*$; (2) $I \subset I^*$;

(3) If $I \subset J$, then $I^* \subset J^*$; (4) $(I^*)^* = I^*$.

Let $*$ be a semistar-operation on R. If, for all finitely generated non-zero fractional ideals J_1, J_2, I with I regular, $(IJ_1)^* \subset (IJ_2)^*$ implies $J_1^* \subset J_2^*$, then $*$ is called an e.a.b. semistar-operation on R.

Let $f = \sum_1^n a_i X^{s_i} \in R[X; S]$, where $s_i \neq s_j$ for $i \neq j$, and $a_i \neq 0$ for each i. Then the ideal $(s_1, \cdots s_n)$ of S is denoted by $e(f)$, and the ideal $(a_1, \cdots, a_n)$ of R is denoted by $c(f)$.

Proposition 1. Let $*$ be a star-operation on a domain D. The following conditions are equivalent:

(1) $*$ is e.a.b.

(2) If $f/g = f'/g'$, where $f, g, f', g' \in D[X]$ with g, g' non-zero, and if $c(f)^* \subset c(g)^*$, then $c(f')^* \subset c(g')^*$.

Proof. Assume that, if $f/g = f'/g'$, where $f, g, f', g' \in D[X]$ with g, g' non-zero, and if $c(f)^* \subset c(g)^*$, then $c(f')^* \subset c(g')^*$. Let I, J_1, J_2 be finitely generated non-zero fractional ideals of D, and assume that $(IJ_1)^* \subset (IJ_2)^*$. We may assume that I, J_1, J_2 are ideals of D. Let $I = (a_0, \cdots, a_n), J_1 = (b_0, \cdots, b_m)$ and $J_2 = (c_0, \cdots, c_l)$. Put $f = \sum a_i X^i, g = \sum b_i X^{i(n+1)}$ and $h = \sum c_i X^{i(n+1)}$. Then $c(fg) = IJ_1, c(fh) = IJ_2$. Since, $(fg)/(fh) = g/h$ and $c(fg)^* \subset c(fh)^*$, we have $c(g)^* \subset c(h)^*$. That is, $J_1^* \subset J_2^*$. Hence $*$ is e.a.b.

In the following, let D be a domain, and let A be a Marot ring with property

Theorem 1. Let $*$ be a star-operation on A. The following conditions are equivalent:

(i) $*$ is e.a.b.

(ii) If $f/g = f'/g'$, where $f, g, f', g' \in A[X; S]$ with g, g' regular, and if $c(f)^* \subset c(g)^*$, then $c(f')^* \subset c(g')^*$.

(2) Let $*$ be a star-opetration on S. The following conditions are equivalent:

(i) $*$ is e.a.b.

(ii) If $f/g = f'/g'$, where $f, g, f', g' \in D[X; S]$ with g, g' non-zero, and if $e(f)^* \subset e(g)^*$, then $e(f')^* \subset e(g')^*$.

(3) Let $*$ be a semistar-operation on A. The following conditions are equivalent:

(i) $*$ is e.a.b.

(ii) If $f/g = f'/g'$, where $f, g, f', g' \in A[X; S]$ with g, g' regular, and if $c(f)^* \subset c(g)^*$, then $c(f')^* \subset c(g')^*$.

(4) Let $*$ be a semistar-operation on S. The following conditions are equivalent:

(i) $*$ is e.a.b.

(ii) If $f/g = f'/g'$, where $f, g, f', g' \in D[X; S]$ with g, g' non-zero, and if $e(f)^* \subset e(g)^*$, then $e(f')^* \subset e(g')^*$.

The proof of Theorem 1 is similar to that of Proposition 1.

Let R be a commutative ring. If every finitely generated regular ideal of R is principal, R is called an r-Bezout ring. If every finitely generated regular ideal of R is invertible, R is called a Prüfer ring. A multiplicative system of R consisting of regular elements is called a regular multiplicative system of R, and a quotient ring of R with respect to a regular multiplicative system is called a regular quotient ring of R.

Let $*$ be a star-operation (resp. semistar-operation) on a g-monoid S. If the set $\{I^* \mid I \text{ is a finitely generated fractional ideal of } S\}$ is a group under the sum $(I_1^*, I_2^*) \longmapsto (I_1^* + I_2^*)^*$, then S is called a Prüfer $*$-multiplication semigroup. Assume that $*$ is an e.a.b. star-operation (resp. semistar-operation) on S, let D be a domain. Then the ring $S_*^D = \{f/g \mid f, g \in D[X; S] - \{0\}$ with

$e(f)^* \subset e(g)^*\} \cup \{0\}$ is called the Kronecker function ring of S with respect to $*$ and D.

Let $*$ be a star-operation (resp. semistar-operation) on R. If the set $\{I^* \mid I$ is a finitely generated regular fractional ideal of $R\}$ is a group under the product $(I_1^*, I_2^*) \longmapsto (I_1^* I_2^*)^*$, then R is called a Prüfer $*$-multiplication ring. Assume that $*$ is an e.a.b. star-operation (resp. semistar-operation) on A. Then the ring $A_*^S = \{f/g \mid f, g \in A[X;S] - \{0\}$ with g regular and $c(f)^* \subset c(g)^*\} \cup \{0\}$ is called the Kronecker function ring of A with respect to $*$ and S.

Let P be a prime ideal of R. The overring $\{x \in L \mid sx \in R$ for some $s \in R - P\}$ of R is denoted by $R_{[P]}$.

Theorem 2. Let $*$ be an e.a.b. star-operation on A, and let $T = \{g \mid g$ is a regular element of $A[X;S]$ with $c(g)^* = A\}$. Then the following conditions are equivalent:

(0) $A[X;S]_T$ is a Prüfer ring.

(1) A is a Prüfer $*$-multiplication ring.

(2) $A[X;S]_T = A_*^S$.

(3) $A[X;S]_T$ is an r-Bezout ring.

(4) Each regular prime ideal of $A[X;S]_T$ is the extension of a prime ideal of A.

(5) A_*^S is a regular quotient ring of $A[X;S]$.

(6) Each prime ideal of $A[X;S]_T$ is the contraction of a prime ideal of A_*^S.

(7) Each regular prime ideal of $A[X;S]_T$ is the contraction of a prime ideal of A_*^S.

(8) Each valuation overring of A_*^S is of the form $A[X;S]_{[PA[X;S]]}$, where P is a prime ideal of A such that $A_{[P]}$ is a valuation overring of A.

(9) A_*^S is a flat $A[X;S]$-module.

Moreover, thre exists a Prüfer Marot ring A with property (A) which satisfies the following conditions: Let $*$ be any e.a.b. $*$-operation on A. Then there exists a prime ideal of $A[X;\mathbf{Z}_0]_T$ which is not the extension of a prime ideal of A, where $\mathbf{Z}_0$ is the g-monoid of non-negative integers.

For the proof of equivalence of (0) $\sim$ (9) we confer [M3, Propositions 3.1 and 3.9 and Theorem 3.7]. Let k be a field, let $X_1, X_2, \cdots, Y_1, Y_2, \cdots$ be indeterminates, and let D_0 be a Prüfer domain. Let $R = k[[X_1, X_2, \cdots, Y_1, Y_2, \cdots]]_1/(X_iX_j, Y_iY_j \mid i \neq j)$, and let $A = R \oplus D_0$, where $k[[X_1, X_2, \cdots, Y_1, Y_2, \cdots]]_1 = \bigcup_{n=1}^{\infty} k[[X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n]]$, and $(X_iX_j, Y_iY_j \mid i \neq j)$ is the ideal of $k[[X_1, X_2, \cdots, Y_1, Y_2, \cdots]]_1$ generated by the subset $\{X_iX_j, Y_iY_j \mid i \neq j\}$. Then A is such a ring (cf. [M1, Theorem (1.3)]).

A similar result to Theorem 2 holds for semistar-operations on the ring A as follows.

Theorem 3. Let $*$ be an e.a.b. semistar-operation on A, and let $W = \{g \mid g$ is a regular element of $A^*[S]$ such that $c(g)^* = A^*\}$. Then the following conditions are equivalent:

(0) $A^*[X; S]_W$ is a Prüfer ring.

(1) A is a Prüfer $*$-multiplication ring.

(2) $A^*[X; S]_W$ coincides with the Kronecker function ring A_*^S of A with respect to $*$ and S.

(3) $A^*[X; S]_W$ is an r-Bezout ring.

(4) Each regular prime ideal of $A^*[X; S]_W$ is the extension of a prime ideal of A^*.

(5) A_*^S is a regular quotient ring of $A^*[X; S]$.

(6) Each prime ideal of $A^*[X; S]_W$ is the contraction of a prime ideal of A_*^S.

(7) Each regular prime ideal of $A^*[X; S]_W$ is the contraction of a prime ideal of A_*^S.

(8) Each valuation overring of A_*^S is of the form $A^*[X; S]_{[QA^*[X;S]]}$, where Q is a prime ideal of A^* such that $(A^*)_{[Q]}$ is a valuation overring of A^*.

(9) A_*^S is a flat $A^*[X; S]$-module.

For the proof we confer [M3, Propositions 3.2, 3.8 and 3.9].

Theorem 4. Let D be a domain, and let $*$ be an e.a.b. star-operation on a g-monoid S, and let $T = \{g \mid g$ is a non-zero element of $D[X; S]$ with $e(g)^* = S\}$. The following conditions are equivalent:

(0) $D[X; S]_T$ is a Prüfer ring.

(1) S is a Prüfer $*$-multiplication semigroup.

(2) $D[X; S]_T$ coincides with the Kronecker function ring S_*^D of S with respect

to $*$ and D.

(3) $D[X;S]_T$ is a Bezout ring.

(4) Each prime ideal of $D[X;S]_T$ is the extension of a prime ideal of S.

(5) S_*^D is a quotient ring of $D[X;S]$.

(6) Each prime ideal of $D[X;S]_T$ is the contraction of a prime ideal of S_*^D.

(7) Each valuation overring of S_*^D is of the form $D[X;S]_{PD[X;S]}$, where P is a prime ideal of S such that S_P is a valuation oversemigroup of S.

(8) S_*^D is a flat $D[X;S]$-module.

For the proof we confer [MS, Theorems 8 and 25].

A similar result to Theorem 4 holds for semistar-operations on S.

Theorem 5. Let $*$ be an e.a.b. semistar-operation on S, and let $W = \{g \mid g$ is a non-zero element of $D[X;S^*]$ such that $\mathrm{e}(g)^* = S^*\}$. The following conditions are equivalent:

(0) $D[X;S^*]_W$ is a Prüfer ring.

(1) S is a Prüfer $*$-multiplication semigroup.

(2) $D[X;S^*]_W = S_*^D$.

(3) $D[X;S^*]_W$ is a Bezout ring.

(4) Each prime ideal of $D[X;S^*]_W$ is the extension of a prime ideal of S^*.

(5) S_*^D is a quotient ring of $D[X;S^*]$.

(6) Each prime ideal of $D[X;S^*]_W$ is the contraction of a prime ideal of S_*^D.

(7) Each valuation overring of S_*^D is of the form $D[X;S^*]_{QD[X;S^*]}$, where Q is a prime ideal of S^* such that $(S^*)_Q$ is a valuation oversemigroup of S^*.

(8) S_*^D is a flat $D[X;S^*]$-module.

For the proof we confer [M2, Proposition 4 and Theorem 23].

Appendix

Theorem. Let S be a g-monoid, and let T be an extension semigroup. If T is a Noetherian semigroup, and if T is a finitely generated S-module, then S is a Noetherian semigroup.

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